

THE PERFECT LOCAL T_b THEOREM AND TWISTED MARTINGALE TRANSFORMS

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ABSTRACT. A local T_b Theorem provides a flexible framework for proving the boundedness of a Calderón-Zygmund operator T . One needs only boundedness of the operator T on systems of locally pseudo-accretive functions $\{b_Q\}$, indexed by cubes. We give a new proof of this Theorem in the setting of perfect (dyadic) models of Calderón-Zygmund operators, imposing integrability conditions on the b_Q functions that are the weakest possible. The proof is a simple direct argument, based upon an inequality for transforms of so-called twisted martingale differences, which has been noted by Auscher-Routin.

1. INTRODUCTION

An operator T is said to be a *perfect Calderón-Zygmund operator* if it satisfies these conditions. There is a kernel $K(x, y)$ so that

$$\langle Tf, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K(x, y) f(y) g(x) \, dy dx$$

for all f, g that are smooth, compactly supported, and the closure of the supports of f and g do not intersect. The kernel $K(x, y)$ is assumed to satisfy the size condition

$$|K(x, y)| \leq \frac{1}{|x - y|^n}$$

and it satisfies the following strong smoothness condition. For any two disjoint dyadic cubes P, Q , $K(x, y)$ is constant on $P \times Q$. The implication of this property, used repeatedly, is this: If f is supported on P and g is supported on Q , and at least one of f and g have integral zero, then $\langle Tf, g \rangle = 0$.

We are interested in bounded Calderón-Zygmund operators, so we set \mathbf{T} to be the norm of T on $L^2(\mathbb{R}^n)$, namely \mathbf{T} is the best constant in the inequality

$$|\langle Tf, g \rangle| \leq \mathbf{T} \|f\|_2 \|g\|_2.$$

It is well known that this inequality extends to the form $|\langle Tf, g \rangle| \lesssim \mathbf{T} \|f\|_p \|g\|_{p'}$, where $1 < p < \infty$ and $1/p + 1/p' = 1$.

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The celebrated T1 Theorem of David and Journé [8] gives a beautiful characterization of the bounded Calderón-Zygmund operators. It was the powerful observation of Michael Christ [6] that a weakening of the T1 Theorem, to a so-called Tb formulation, can have wide-ranging implications. Christ himself addressed analytic capacity, and it was this perspective that was crucial to the solution of the Kato square root problem [2, 10]. Our focus is on the local Tb, in the dyadic model, as promoted in [1]. This is the usual definition of systems of accretive functions.

1.1. Definition. Fix $1 < p < \infty$. A collection of functions $\{b_Q : Q \in \mathcal{D}\}$ is called a *system of p-accretive functions with constant* $1 < A$ if these conditions hold for each dyadic cube $Q \in \mathcal{D}$.

- (1) b_Q is supported on Q and $\int_Q b_Q(x) dx = |Q|$.
- (2) $\|b_Q\|_p \leq A|Q|^{1/p}$.

In the Theorem below p_1, p_2 are not related by duality; for instance it is allowed that $1 < p_1, p_2 < 2$.

1.2. Theorem. For fixed constants A and T_{loc} , this holds. Suppose that T is a perfect dyadic Calderón-Zygmund operator and, for $1 < p_1, p_2 < \infty$, there are systems $\{b_Q^j\}$ of p_j -accretive functions with constant A , so that

$$\int_Q |Tb_Q^1|^{p'_2} dx \leq T_{\text{loc}}^{p'_2} |Q|, \quad \int_Q |T^*b_Q^2|^{p'_1} dx \leq T_{\text{loc}}^{p'_1} |Q|.$$

Then, T extends to a bounded operator on L^2 , and moreover, $T \lesssim_{A, p_1, p_2} 1 + T_{\text{loc}}$.

This is a known result, [1, Theorem 6.8]. Auscher and Routin [3, Section 8] have recently devised a proof closely related to this one.

Martingale transform inequality for twisted differences play the central role. These inequalities have also been used by Auscher-Routin [3, Section 5]. The direct proof of the the Theorem proceeds by standard reductions, and construction of stopping cubes from the local Tb hypotheses, and a brief additional argument. A highlight is a simple appeal to the local Tb hypothesis and the martingale transform inequality. Compare to [3, Estimate for $\langle f, V_{1,2}g \rangle$].

Relevant history, and indications of the utility of Tb theorems can be found in surveys by S. Hofmann [11, 12]. See in particular [11, §3.3.1], where the extension of the Theorem above to the continuous case is specifically mentioned. The perfect case is of course very special, still the argument in [1] has been influential, although the task of lifting the proof therein to the continuous case has not proven to be easy. Auscher and Yang [4] succeeded in extending the Theorem above to the continuous case, with the duality assumption on p_1 and p_2 but the argument is an indirect reduction to the perfect case. This is less desirable, due to the interest in local Tb theorems more general settings, such as the setting of homogeneous spaces, as in Auscher and Routin [3]. The latter paper employs the Belykin-Coifman-Rohklin algorithm, see [5, 9]. The latter paper addresses the the case where $1/p_1 + 1/p_2 > 1$, but additional hypotheses are needed, and their nature is still unresolved. One can also consult Hytönen-Martikainen [19, 20] for another general approach to local Tb Theorem in non-homogeneous and upper doubling settings, although in the setting where duality is imposed. A local Tb Theorem in a vector-valued setting, with strong conditions on accretive functions, is considered in [21]. Salamone [22] also studies the dyadic Tb Theorem.

Notation: For any cube Q , $\langle f \rangle_Q := |Q|^{-1} \int_Q f \, dx$, and $\ell Q = |Q|^{1/n}$ is the side length of the cube. $A \lesssim B$ means that $A \leq C \cdot B$, where C is an unspecified constant independent of A and B .

2. THE MARTINGALE TRANSFORM INEQUALITY

The classical martingale transform inequality is this. For all constants satisfying $|\varepsilon_Q| \leq 1$,

$$(2.1) \quad \left\| \sum_{Q \in \mathcal{D}} \varepsilon_Q \sum_{Q' \in \text{ch}(Q)} \{ \langle f \rangle_{Q'} - \langle f \rangle_Q \} \mathbf{1}_{Q'} \right\|_q \lesssim \|f\|_q, \quad 1 < q < \infty.$$

A variant is stated in Theorem 2.3, and it is essential to the subsequent arguments. This section can be read independently of the rest of the paper. Above, and for the remainder of the paper, $\langle f \rangle_Q := |Q|^{-1} \int_Q f \, dx$ is the average of f on cube Q .

Fix a function b supported on a dyadic cube S_0 , satisfying $\int b \, dx = |S_0|$, and $\|b\|_p \leq A|S_0|^{1/p}$, where $1 < p < \infty$ is fixed. Let \mathcal{T}' be the maximal dyadic cubes $T \subset S_0$ which meet either one of these conditions with $\delta \in (0, 1)$:

$$(2.2) \quad \left| \int_T b \, dx \right| \leq \delta |T| \quad \text{or} \quad \int_T |b|^p \, dx \geq \delta^{-1} A^p |T|.$$

We will consider a fixed but arbitrary family \mathcal{T} of disjoint dyadic cubes in S_0 , the ‘terminal cubes’, and our estimates are not allowed depend upon this family. Moreover, we require that $T' \subset T \in \mathcal{T}$ if $T' \in \mathcal{T}'$. To each terminal cube T , we have a function b_T supported on T , and satisfying $\int b_T \, dx = |T|$ and $\|b_T\|_p \leq A|T|^{1/p}$.

Let \mathcal{Q} be all dyadic cubes, contained in S_0 , but not contained in any $T \in \mathcal{T}$. Define

$$\Delta_Q f := \sum_{Q' \in \text{ch}(Q)} \left[\frac{\langle f \rangle_{Q'}}{\langle b_{Q'} \rangle_{Q'}} b_{Q'} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} b \right] \mathbf{1}_{Q'}, \quad Q \in \mathcal{Q},$$

where we set $b_{Q'} = b$ if $Q' \notin \mathcal{T}$ and otherwise, $b_{Q'}$ is defined as above. We refer to these as the *twisted martingale differences*.

2.3. Theorem. *This inequality holds for all selection of constants $|\varepsilon_Q| \leq 1$.*

$$\left\| \sum_{Q \in \mathcal{Q}} \varepsilon_Q \Delta_Q f \right\|_p \lesssim \|f\|_p,$$

where $1 < p < \infty$ is the exponent associated with the function b .

This Theorem and Theorem 2.5 below are contained in [3, Lemma 5.3]. A randomized version of this Theorem in a vector-valued context is proven in [21, Section 4]. And, the more common square function variant is well-known. We give a somewhat different proof, in the spirit of completeness, since we view the inequality as fundamental to the Tb theorems.

We need the following preparation. In the sum below, we do not sum over the children which are terminal cubes, and we do not multiply by the b functions, and so we refer to these as the

half-twisted differences.

$$(2.4) \quad D_Q f := \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \left[\frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'}} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \right] \mathbf{1}_{Q'}, \quad Q \in \mathcal{Q}.$$

The following universal estimate holds in Lebesgue measure.

2.5. Theorem. *These inequalities hold for all selection of constants $|\varepsilon_Q| \leq 1$.*

$$\left\| \sum_{Q \in \mathcal{Q}} \varepsilon_Q D_Q f \right\|_q \lesssim \|f\|_q, \quad 1 < q < \infty.$$

Proof. It is important to note that this operator is, in fact, a constant multiple of a perfect Calderón-Zygmund operator. It therefore suffices to verify the conditions of the T1 Theorem, but this is not convenient to do directly. Instead, we write the operator as a sum of three perfect Calderón-Zygmund operators. In verifying the T1 conditions for these operators, we use the \mathbf{T}_{weak} constant, testing the L^1 and/or L^p norm of $\mathbf{T}\mathbf{1}_F$ and $\mathbf{T}^*\mathbf{1}_F$ for cubes F . Recall that $b \in L^p$.

For cube $Q \in \mathcal{Q}$ with child $Q' \notin \mathcal{T}$, we write

$$\begin{aligned} \frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'}} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} &= \left\{ \frac{\langle f \rangle_{Q'}}{\langle b \rangle_Q} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \right\} + \left\{ \frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'}} - \frac{\langle f \rangle_{Q'}}{\langle b \rangle_Q} \right\} \\ &= \left\{ \frac{\langle f \rangle_{Q'}}{\langle b \rangle_Q} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \right\} + \{ \langle b \rangle_Q - \langle b \rangle_{Q'} \} \frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'} \langle b \rangle_Q} \\ (2.6) \quad &= \left\{ \frac{\langle f \rangle_{Q'}}{\langle b \rangle_Q} - \frac{\langle f \rangle_Q}{\langle b \rangle_Q} \right\} \end{aligned}$$

$$(2.7) \quad + \{ \langle b \rangle_Q - \langle b \rangle_{Q'} \} \frac{\langle f \rangle_{Q'}}{\langle b \rangle_Q^2}$$

$$(2.8) \quad + \{ \langle b \rangle_Q - \langle b \rangle_{Q'} \}^2 \frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'} \langle b \rangle_Q^2}$$

This gives us three sums to bound. Keep in mind that the averages of b that occur are bounded from above and below by failure of (2.2). In the first two expressions, the denominator is only a function of Q , while in the third, it depends upon the child Q' , with however the square on the difference on b . The first term gives rise to a classical martingale difference on f , the second a martingale difference on b , and the third, a square function of a martingale difference on b .

Let us observe that

$$(2.9) \quad \left\| \sum_{Q \in \mathcal{Q}} \varepsilon_Q \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \{ \langle f \rangle_{Q'} - \langle f \rangle_Q \} \mathbf{1}_{Q'} \right\|_q \lesssim \|f\|_q, \quad 1 < q < \infty.$$

Indeed, this is a consequence of the classical martingale transform inequality (2.1) and maximal function estimates in the disjoint family of missing terminal cubes. The desired estimate for the sum associated with terms (2.6) follows from this.

An estimate for the sum associated with term (2.7) is clearly a consequence of inequality,

$$(2.10) \quad \left\| \sum_{Q \in \mathcal{Q}} \varepsilon_Q \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \{ \langle b \rangle_{Q'} - \langle b \rangle_Q \} \langle f \rangle_{Q'} \mathbf{1}_{Q'} \right\|_q \lesssim \mathbf{A} \|f\|_q, \quad 1 < q < \infty.$$

Denote the linear operator on the left hand side by Π . After normalizing with a constant $c_{n,\delta} \mathbf{A}^{-1}$, we are looking for L^q -norm estimates for a symmetric perfect Calderón-Zygmund operator. It is classical that it suffices to verify inequality,

$$(2.11) \quad \|\Pi \mathbf{1}_F\|_{L^1(F)} \lesssim |F|,$$

where F is a dyadic cube. In order to do this, let us write

$$(2.12) \quad \Pi \mathbf{1}_F = \left\{ \sum_{\substack{Q \in \mathcal{Q} \\ Q \supsetneq F}} + \sum_{\substack{Q \in \mathcal{Q} \\ Q \subset F}} \right\} \varepsilon_Q \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \{ \langle b \rangle_{Q'} - \langle b \rangle_Q \} \langle \mathbf{1}_F \rangle_{Q'} \mathbf{1}_{Q'}.$$

Using Minkowski inequality and the trivial estimate $|\langle \mathbf{1}_F \rangle_{Q'}| \leq |F| |Q'|^{-1}$, we find that L^p -norm of the first series is bounded by $c_n \delta^{-1} \mathbf{A} |F|^{1/p}$. Concerning the second series, we can clearly assume that $Q \subset F$ for some $Q \in \mathcal{Q}$. Let us denote by R the maximal cube in \mathcal{Q} , contained in F . Assuming $Q \ni Q \subset F$ and Q' is a child of Q , then $\langle \mathbf{1}_F \rangle_{Q'} = 1$, $\langle b \rangle_{Q'} = \langle b \mathbf{1}_R \rangle_{Q'}$, and likewise $\langle b \rangle_Q = \langle b \mathbf{1}_R \rangle_Q$. By inequality (2.9), setting $\varepsilon_Q = 0$ if $Q \ni F$, the L^p -norm of the second series in (2.12) is bounded by $\|b \mathbf{1}_R\|_p$ which, in turn, is bounded by $\delta^{-1/p} \mathbf{A} |F|^{1/p}$. This concludes the proof of inequality (2.11) and, as a consequence, we obtain inequality (2.10).

It remains to estimate the sum associated with term (2.8). Namely, we need the following inequality,

$$\left\| \sum_{Q \in \mathcal{Q}} \varepsilon_Q \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \{ \langle b \rangle_{Q'} - \langle b \rangle_Q \}^2 \frac{\langle f \rangle_{Q'}}{\langle b \rangle_{Q'} \langle b \rangle_Q^2} \mathbf{1}_{Q'} \right\|_q \lesssim \mathbf{A} \|f\|_q, \quad 1 < q < \infty.$$

Denote the linear operator on the left hand side by Π . Again, after a normalization by $c_{n,\delta} \mathbf{A}^{-2}$, we are looking for L^q estimates of perfect Calderón-Zygmund operator. By symmetry of Π , it suffices to verify that

$$(2.13) \quad \|\Pi \mathbf{1}_F\|_{L^1(F)} \lesssim |F|,$$

where F is a dyadic cube. In order to verify this inequality, we split the series defining $\Pi \mathbf{1}_F$ in two parts as above, one with cubes $Q \ni F$ and the other with cubes $Q \subset F$. Reasoning as above, we find that the L^2 -norm of the first series is bounded by $c_{n,\delta} \mathbf{A}^2 |F|^{1/2}$. The second series to estimate is

$$\left\| \sum_{\substack{Q \in \mathcal{Q} \\ Q \subset F}} \varepsilon_Q \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \{ \langle b \rangle_{Q'} - \langle b \rangle_Q \}^2 \frac{\langle \mathbf{1}_F \rangle_{Q'}}{\langle b \rangle_{Q'} \langle b \rangle_Q^2} \mathbf{1}_{Q'} \right\|_1.$$

Using the fact that $\langle \mathbf{1}_F \rangle_{Q'} = 1$ if $Q \subset F$ and Q' is a child of Q , yields the upper bound

$$c_\delta \sum_{Q \in \mathcal{Q}} \sum_{Q' \in \text{ch}(Q) \setminus \mathcal{T}} \int_{\mathbf{R}^n} \left| \{ \langle b \rangle_{Q'} - \langle b \rangle_Q \} \langle \mathbf{1}_F \rangle_{Q'} \mathbf{1}_{Q'}(x) \right|^2 dx,$$

where we have also relaxed the summation condition using positivity of the summands. This upper bound, in turn, is bounded by a constant multiple of $\mathbf{A}^2 \|\mathbf{1}_F\|_2^2 = \mathbf{A}^2 |F|$ – a consequence of a square function estimate arising from randomization of inequality (2.10) with $q = 2$ by taking ε_Q 's to be Rademacher random variables. This completes the proof of inequality (2.13) and, consequently, the proof of theorem. \square

We have the following easier proposition.

2.14. Proposition. *This inequality holds for all selection of constants $|\varepsilon_Q| \leq 1$.*

$$\left\| \sum_{T \in \mathcal{T}} \varepsilon_{T^{(1)}} \langle f \rangle_T \mathbf{1}_T \right\|_q + \left\| \sum_{T \in \mathcal{T}} \varepsilon_{T^{(1)}} \frac{\langle f \rangle_{T^{(1)}}}{\langle b \rangle_{T^{(1)}}} \mathbf{1}_T \right\|_q \lesssim \|f\|_q, \quad 1 < q < \infty.$$

Proof. By disjointness of terminal cubes and the estimate $|\langle b \rangle_{T^{(1)}}| \geq \delta$ for $T \in \mathcal{T}$,

$$\begin{aligned} \left\| \sum_{T \in \mathcal{T}} \varepsilon_{T^{(1)}} \frac{\langle f \rangle_{T^{(1)}}}{\langle b \rangle_{T^{(1)}}} \mathbf{1}_T \right\|_q &\leq \delta^{-1} \left(\sum_{T \in \mathcal{T}} \int_{\mathbf{R}^n} |\langle f \rangle_{T^{(1)}} \mathbf{1}_T(x)|^q dx \right)^{1/q} \\ &\lesssim \delta^{-1} \|Mf\|_q \lesssim \delta^{-1} \|f\|_q. \end{aligned}$$

The other term is estimated in a similar manner. \square

Proof of Theorem 2.3. Let us set $Bf := \sum_{Q \in \mathcal{Q}} \varepsilon_Q D_Q f$, and observe that

$$(2.15) \quad \sum_{Q \in \mathcal{Q}} \varepsilon_Q D_Q f = Bf \cdot b + \sum_{T \in \mathcal{T}} \varepsilon_{T^{(1)}} \langle f \rangle_T \mathbf{1}_T \cdot \sum_{T \in \mathcal{T}} b_T - \sum_{T \in \mathcal{T}} \varepsilon_{T^{(1)}} \frac{\langle f \rangle_{T^{(1)}}}{\langle b \rangle_{T^{(1)}}} \mathbf{1}_T \cdot b.$$

Consider the events $E_\lambda := \{|Bf| \geq \lambda\} \subset S_0$, where $\lambda > 0$. Let $S_{\mathcal{T}} \subset S_0$ be the union of terminal cubes $T \in \mathcal{T}$. By construction of \mathcal{T} , and Lebesgue differentiation Theorem, we have $|b(x)| \leq \delta^{-1} \mathbf{A}$ for almost every $x \in S_0 \setminus S_{\mathcal{T}}$. Hence,

$$\int_{E_\lambda \setminus S_{\mathcal{T}}} |b|^p dx \leq \delta^{-p} \mathbf{A}^p |E_\lambda \setminus S_{\mathcal{T}}|.$$

Observe that Bf is constant on terminal cubes. Let us denote $\mathcal{T}_\lambda := \{T \in \mathcal{T} : |Bf| \geq \lambda \text{ on } T\}$. Since $T^{(1)} \in \mathcal{Q}$ for each terminal cube T ,

$$\begin{aligned} \int_{E_\lambda \cap S_{\mathcal{T}}} |b|^p dx &= \sum_{T \in \mathcal{T}_\lambda} |T| \left(\frac{1}{|T|} \int_T |b|^p dx \right) \\ &\leq 2^n \delta^{-p} \mathbf{A}^p \sum_{T \in \mathcal{T}_\lambda} |T| \leq 2^n \delta^{-p} \mathbf{A}^p |E_\lambda \cap S_{\mathcal{T}}|. \end{aligned}$$

It follows that we can compare Lebesgue measure estimates and estimates with respect to $|b(x)|^p dx$. Namely,

$$\int_{E_\lambda} |b|^p dx \leq 2^n \delta^{-p} \mathbf{A}^p |E_\lambda|.$$

Therefore, by a standard formula and the Lebesgue measure estimates of Theorem 2.5,

$$\int_{S_0} |Bf|^p |b|^p dx = p \int_0^\infty \lambda^{p-1} \int_{E_\lambda} |b|^p dx d\lambda \lesssim A^{3p} \|f\|_p^p.$$

The L^p -norms of the two remaining quantities in the right hand side of (2.15) are estimated in a similar manner, by using Proposition 2.14 and measures $\sum_{T \in \mathcal{T}} |b_T|^p dx$ and $|b(x)|^p dx$ instead. \square

3. THE CORONA

We will work with different subsets of the dyadic grid, and need some notations. Given $Q \in \mathcal{D}$, we denote by $\text{ch}(Q)$ the 2^n dyadic children of Q . Given $\mathcal{S} \subset \mathcal{D}$, we can refer to the \mathcal{S} -children of $S \in \mathcal{S}$: The maximal elements S' of \mathcal{S} that are strictly contained in S . This collection is denoted as $\text{ch}_{\mathcal{S}}(S)$. For a cube $Q \in \mathcal{D}$, that is contained in a cube in \mathcal{S} , we take $\pi_{\mathcal{S}}Q$ to be the \mathcal{S} -parent of Q : The smallest cube in \mathcal{S} that contains Q .

This is the construction of stopping cubes: for a fixed $Q_0 \in \mathcal{D}$, families $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{D}$ are defined as follows. Take the cube Q_0 in \mathcal{S}_1 . In the inductive stage, if $S \in \mathcal{S}_1$, take as members of \mathcal{S}_1 those maximal dyadic descendants Q which meet any one of these several conditions:

- (1) $\left| \int_Q b_S^1 \right| \leq \delta |Q|$.
- (2) $\int_Q |b_S^1|^{p_1} \geq \delta^{-1} A^{p_1} |Q|$.
- (3) $\int_Q |Tb_S^1|^{p_2'} \geq \delta^{-1} T_{\text{loc}}^{p_2'} |Q|$.

A stopping tree \mathcal{S}_2 is then constructed analogously, but using functions $\{b_S^2\}_{S \in \mathcal{D}}$, and exponents p_2 and p_1' in conditions (2) and (3), respectively. Furthermore, in (3) we use T^* instead of T .

If $\delta > 0$ is chosen sufficiently small in the construction above, then there is a constant $\tau \in (0, 1)$ such that

$$(3.1) \quad \sum_{S' \in \text{ch}_{\mathcal{S}_1}(S)} |S'| \leq \tau |S|, \quad S \in \mathcal{S}_1.$$

Here both τ and δ depend on A . It follows from construction of \mathcal{S}_1 and inequality (3.1) – and their counterparts in \mathcal{S}_2 – that \mathcal{S} is a Carleson family of cubes. Namely, there holds

$$\sum_{S \in \mathcal{S}: S \subset Q} |S| \lesssim_\tau |Q|, \quad \mathcal{D} \ni Q \subset Q_0.$$

In the sequel, we suppose that δ is chosen as above.

3.2. Remark. With \mathcal{S} so constructed, many subsequent inequalities have constant that depend upon the values of δ and A . The dependence is not straightforward, and we do not attempt to track it. Frequently, this dependence is even suppressed in the notation, so that in many parts of the argument relative to \mathcal{S} , the symbol ' \lesssim ' should be read as 'the unspecified implied constant depends upon dimension, τ , δ and A , but is otherwise absolute.' Dependencies on other parameters are indicated by subscripts, e.g. $\lesssim_p 1$ means $\lesssim c_p$ where c_p depends on p only.

Following [7], we define the b -adapted conditional expectations and martingale differences, associated with a dyadic cube $Q \subset Q_0$, by

$$E_Q^j h := \frac{\langle h \rangle_Q}{\langle b_{\pi_{S_j} Q}^j \rangle_Q} b_{\pi_{S_j} Q}^j \mathbf{1}_Q, \quad \Delta_Q^j h := \sum_{Q' \in \text{ch}(Q)} [E_{Q'}^j h - E_Q^j h] \cdot \mathbf{1}_{Q'}.$$

Observe that the functions $\Delta_Q^j h$ have mean zero in \mathbf{R}^n , and they are supported in the cube Q .

The following Lemma is proven like Lemma 3.5 in [20], with obvious modifications.

3.3. Lemma. *Let $h \in L^{p_j}(\mathbf{R}^n)$, $j \in \{1, 2\}$. Then, there holds pointwise and in $L^{p_j}(\mathbf{R}^n)$,*

$$h \mathbf{1}_S = E_S^j h + \sum_{Q: Q \subset S} \Delta_Q^j h, \quad S \subset Q_0.$$

We will suppress the notation by denoting $E_Q^j = E_Q$ and $\Delta_Q^j = \Delta_Q$, $j \in \{1, 2\}$.

By a well-known reduction from the T1 Theorem, it suffices to show that

$$|\langle Tf, g \rangle| \lesssim (1 + \mathbf{T}_{\text{loc}}) |Q_0|,$$

where f and g are measurable functions with the property $|f| = |g| = \mathbf{1}_{Q_0}$, [1, 3, 11]. By Lemma 3.3, the expansion of the bilinear form is

$$\langle Tf, g \rangle = \langle TE_{Q_0} f, g \rangle + \langle T \sum_{P: P \subset Q_0} \Delta_P f, E_{Q_0} g \rangle + \sum_{P, Q: P \cup Q \subset Q_0} \langle T \Delta_P f, \Delta_Q g \rangle.$$

Using the assumptions, it is straightforward to verify that

$$|\langle TE_{Q_0} f, g \rangle| \leq \mathbf{T}_{\text{loc}} |Q_0|^{1/p'_2} |Q_0|^{1/p_2}$$

and, by using also Lemma 3.3, that

$$|\langle T \sum_{P: P \subset Q_0} \Delta_P f, E_{Q_0} g \rangle| \leq A \mathbf{T}_{\text{loc}} |Q_0|^{1/p_1} |Q_0|^{1/p'_1}.$$

Since $1/p_1 + 1/p'_1 = 1 = 1/p_2 + 1/p'_2$, we are left with estimating the main term

$$\sum_{P, Q} \langle T \Delta_P f, \Delta_Q g \rangle = \left\{ \sum_{P, Q: \text{ell} P < \text{ell} Q} + \sum_{P, Q: \text{ell} P = \text{ell} Q} + \sum_{P, Q: \text{ell} P > \text{ell} Q} \right\} \langle T \Delta_P f, \Delta_Q g \rangle,$$

where all the summations are restricted to dyadic cubes P, Q contained in Q_0 . As is standard, we will assume that T has kernel K which is perfect, and $K(x, y)$ is identically zero for $|x - y|$ sufficiently small. In particular, the sum above can be taken over a finite collection of P, Q . We will rearrange the sum, as is convenient for us. All bounds are independent of the this last assumption on the kernel.

We will show that the second and third forms above obey the desired estimate, and by duality the same will hold for the first term. To state this otherwise, it suffices to consider the two forms

$$B_{\text{above}}(f, g) := \sum_{P, Q: P \supseteq Q} \langle T(\Delta_P f), \Delta_Q g \rangle,$$

$$B_=(f, g) := \sum_{P, Q : P=Q} \langle T\Delta_P f, \Delta_Q g \rangle ,$$

where the summations are restricted to cubes P, Q that are contained in Q_0 . We will suppress this notationwise also in the sequel, but this fact will be used nevertheless. It is noteworthy that the martingale transform inequality is decisive in estimating both of these terms.

4. THE TERM B_{above}

We address a book keeping issue. For cube P set

$$\tilde{\Delta}_P f := \sum_{P' \in \text{ch}(P) \setminus \mathcal{S}_1} \frac{\langle f \rangle_{P'}}{\langle b_{\pi_{\mathcal{S}_1} P'}^1 \rangle_{P'}} \mathbf{1}_{P'} - \frac{\langle f \rangle_P}{\langle b_{\pi_{\mathcal{S}_1} P}^1 \rangle_P} \mathbf{1}_P .$$

This is closely related to a half-twisted martingale difference associated with P . It suffices to show that for any $S \in \mathcal{S}_1$

$$\left| \mathbf{1}_{\{S \neq Q_0\}} \cdot \langle f \rangle_S \sum_{Q : Q \subset S} \langle T b_S^1, \Delta_Q g \rangle + \sum_{P : \pi_{\mathcal{S}_1} P = S} \sum_{Q \subsetneq P} \langle T(b_S^1 \tilde{\Delta}_P f), \Delta_Q g \rangle \right| \lesssim T_{\text{loc}} |S| .$$

Indeed, the sum of the left-hand side over $S \in \mathcal{S}_1$ equals $B_{\text{above}}(f, g)$, and the collection \mathcal{S}_1 is a Carleson sequence of cubes. The terms involving f only depend upon b_S^1 , which is a convenience.

The first term is easy to estimate. The twisted martingale differences on g telescope, so that

$$\sum_{Q \subsetneq S} \Delta_Q g = g \mathbf{1}_S - \frac{\langle g \rangle_S}{\langle b_{\pi_{\mathcal{S}_2} S}^2 \rangle_S} b_{\pi_{\mathcal{S}_2} S}^2 \mathbf{1}_S .$$

The ratio of averages is controlled, by construction. By the local Tb assumptions and construction,

$$|\langle T b_S^1, g \mathbf{1}_S \rangle| + |\langle T b_S^1, b_{\pi_{\mathcal{S}_2} S}^2 \mathbf{1}_S \rangle| \lesssim T_{\text{loc}} |S| .$$

For the second term, the twisted martingale transform is the decisive point. For pairs of cubes $Q \subsetneq P$, let P_Q denote the child of P that contains Q . The property of T being perfect, and $\Delta_Q g$ having integral zero, allows us to write

$$\begin{aligned} \langle T(b_S^1 \tilde{\Delta}_P f), \Delta_Q g \rangle &= \langle T(b_S^1 \tilde{\Delta}_P f \cdot \mathbf{1}_{P_Q}), \Delta_Q g \rangle \\ &= \langle \tilde{\Delta}_P f \rangle_{P_Q} \langle T(b_S^1 \mathbf{1}_{P_Q}), \Delta_Q g \rangle \\ &= \langle \tilde{\Delta}_P f \rangle_{P_Q} \langle T b_S^1, \Delta_Q g \rangle . \end{aligned}$$

We have first restricted the argument of T to the cube P_Q , pulled out the constant value of $\tilde{\Delta}_P f$ on that cube, and finally extended the argument of T to the entire cube S .

Now, fix $Q \subsetneq S$, and define a constant ε_Q by the formula

$$\varepsilon_Q := \sum_{\substack{P : P \supsetneq Q \\ \pi_{\mathcal{S}_1} P = S}} \langle \tilde{\Delta}_P f \rangle_{P_Q} .$$

These numbers are bounded by a constant, since the sum is telescoping, and equals the difference of two b -averages of f (or a single average, in case of $\pi_{S_1} Q \subsetneq S$), which are bounded. We can make a direct appeal to the local Tb hypothesis.

$$\begin{aligned} \left| \sum_{P: \pi_{S_1} P=S} \sum_{Q \subsetneq P} \langle \tilde{\Delta}_P f \rangle_{P_Q} \langle \text{Tb}_S^1, \Delta_Q g \rangle \right| &= \left| \sum_{Q \subsetneq S} \langle \text{Tb}_S^1, \varepsilon_Q \Delta_Q g \rangle \right| \\ &\leq \mathbf{T}_{\text{loc}} |S|^{1/p'_2} \left\| \sum_{Q \subsetneq S} \varepsilon_Q \Delta_Q g \right\|_{p_2} \lesssim \mathbf{T}_{\text{loc}} |S|. \end{aligned}$$

The twisted martingale transform inequality and the construction provide the last inequality.

Indeed, let S_2 be the S_2 parent of S , and set $\mathcal{R}_1 := \{S_2\}$. Let \mathcal{R}_2 be the S_2 children of S_2 strictly contained in S , and inductively set \mathcal{R}_{k+1} to the S_2 children of cubes $R \in \mathcal{R}_k$. Each function below is a twisted martingale transform of g

$$\gamma_R := \sum_{\substack{Q: \pi_{S_2} Q=R \\ Q \subsetneq S}} \varepsilon_Q \Delta_Q g, \quad R \in \bigcup_{k=1}^{\infty} \mathcal{R}_k.$$

There holds $\|\gamma_R\|_{p_2} \lesssim |R \cap S|^{1/p_2}$, by the martingale transform inequality and the fact that g is a bounded function. Moreover, from (3.1), it follows that

$$\sum_{R \in \mathcal{R}_k} |R \cap S| \lesssim \tau^k |S|,$$

where $0 < \tau < 1$ is fixed. Hence,

$$\begin{aligned} \left\| \sum_{Q \subsetneq S} \varepsilon_Q \Delta_Q g \right\|_{p_2}^{p_2} &= \left\| \sum_{k=1}^{\infty} \sum_{R \in \mathcal{R}_k} k^{-1+1} \gamma_R \right\|_{p_2}^{p_2} \\ &\lesssim \sum_{k=1}^{\infty} k^{p_2} \sum_{R \in \mathcal{R}_k} \|\gamma_R\|_{p_2}^{p_2} \\ &\lesssim |S| \sum_{k=1}^{\infty} k^{p_2} \tau^k \lesssim |S|. \end{aligned}$$

This completes the analysis of the above form.

5. THE DIAGONAL TERM

One can compare this argument to that of [3, Section 8.1]. Before beginning the main thrust of the argument, a particular consequence of the martingale transform inequality is needed. Using the notation of Theorem 2.5, set for $j = 1, 2$,

$$\square_Q^j h := |D_Q^j h| + \begin{cases} 1_Q & \text{a child of } Q \text{ is in } S_j \\ 0 & \text{otherwise} \end{cases}$$

where D_Q^j is defined as in (2.4), with $\mathcal{S}_0^j := \pi_{\mathcal{S}_j} S$, terminal cubes $\mathcal{T}^j := \text{ch}_{\mathcal{S}_j}(S_0^j)$, and function $b^j := b_{\mathcal{S}_0^j}^j$. Note that second summand accounts for the missing terminal cubes in definition of D_Q^j .

By a randomization argument, the half-twisted inequality of Theorem 2.5, and the Carleson measure property of the cubes, there holds

$$\left\| \left[\sum_{Q: Q \subset Q_0} (\square_Q^1 f)^2 \right]^{1/2} \right\|_q \lesssim |Q_0|^{1/q}, \quad 1 < q < \infty.$$

The same inequality holds for g .

To control the diagonal term, it therefore suffices to show that

$$|\langle T\Delta_Q f, \Delta_Q g \rangle| \lesssim (1 + \mathbf{T}_{\text{loc}}) \sum_{Q^1, Q^2 \in \text{ch}(Q)} \langle \square_Q^1 f \rangle_{Q^1} |Q| \langle \square_Q^2 g \rangle_{Q^2}, \quad Q \subset Q_0.$$

For cube Q , and child Q^1 of Q , $\Delta_Q f \mathbf{1}_{Q^1}$ is either a multiple of $b_{\pi_{\mathcal{S}_1} Q}^1 \mathbf{1}_{Q^1}$, or, in the exclusive case that Q^1 is also a stopping cube, a linear combination of this function and $b_{Q^1}^1$. In both cases, the coefficients in the linear combination are dominated by a constant times $\langle \square_Q^1 f \rangle_{Q^1}$. Therefore, the control of the term above follows from this Lemma.

5.1. Lemma. *Suppose that $Q \subset Q_0$. Then, if Q^j is a children of Q and $b^j \in \{b_{\pi_{\mathcal{S}_j} Q}^j, b_{Q^j}^j\}$ with $j = 1, 2$,*

$$|\langle T(b^1 \mathbf{1}_{Q^1}), b^2 \mathbf{1}_{Q^2} \rangle| \lesssim (1 + \mathbf{T}_{\text{loc}}) |Q|.$$

Proof. Let us consider the case $b^j = b_{\pi_{\mathcal{S}_j} Q}^j$, $j = 1, 2$. The other cases are similar but easier. Suppose first that $Q^1 \neq Q^2$. Then, since T is perfect, we see that K is constant on $Q^2 \times Q^1$. Hence, by denoting the midpoint of Q^j by x_{Q^j} ,

$$\begin{aligned} |\langle T(b^1 \mathbf{1}_{Q^1}), b^2 \mathbf{1}_{Q^2} \rangle| &= |K(x_{Q^2}, x_{Q^1})| \cdot \left| \int_{Q^1} b_{\pi_{\mathcal{S}_1} Q}^1(y) dy \right| \cdot \left| \int_{Q^2} b_{\pi_{\mathcal{S}_2} Q}^2(x) dx \right| \\ &\lesssim |Q|. \end{aligned}$$

In the last step we used the kernel size estimate.

Then we suppose that $Q^1 = Q^2$. We let $b_{Q^1}^2$ be the p_2 -accretive function, associated with the cube Q^1 . It suffices to estimate the following terms,

$$(5.2) \quad \left| \langle T(b^1 \mathbf{1}_{Q^1}), \langle b^2 \rangle_{Q^1} b_{Q^1}^2 \rangle \right| + \left| \langle T(b^1 \mathbf{1}_{Q^1}), b^2 \mathbf{1}_{Q^1} - \langle b^2 \rangle_{Q^1} b_{Q^1}^2 \rangle \right|.$$

The first term is bounded by

$$|\langle b^2 \rangle_{Q^1}| \cdot \left| \langle b^1 \mathbf{1}_{Q^1}, T^*(b_{Q^1}^2) \rangle \right| \lesssim \mathbf{T}_{\text{loc}} |Q|.$$

Here we used Hölder's inequality, with both exponents p_2 and p_1 . To estimate the second term in (5.2), the crucial step is to remove the characteristic function $\mathbf{1}_{Q^1}$ from within $T(b^1 \mathbf{1}_{Q^1})$. For

this purpose, let us observe that the function $B^2 := b^2 \mathbf{1}_{Q^1} - \langle b^2 \rangle_{Q^1} b_{Q^1}^2$ is supported on Q^1 and it has zero integral. By assumption that T is perfect,

$$|\langle T(b^1 \mathbf{1}_{Q^1}), B^2 \rangle| = |\langle T(b^1), B^2 \rangle| \lesssim T_{\text{loc}} |Q|.$$

In the last step, we split the dual form in two other forms and use Hölder's inequality, with exponent p_2 , for the individual forms separately. \square

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